RENORMALIZATION GROUP, OPERATOR PRODUCT EXPANSION AND ANOMALOUS SCALING IN MODELS OF PASSIVE TURBULENT ADVECTION

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The field theoretic renormalization group is applied to Kraichnan's model of a passive scalar quantity advected by the Gaussian velocity field with the pair correlation function $\propto \delta(t-t')/k^{d+\varepsilon}$. Inertial-range anomalous scaling for the structure functions and various pair correlators is established as a consequence of the existence in the corresponding operator product expansions of "dangerous" composite operators (powers of the local dissipation rate), whose *negative* critical dimensions determine anomalous exponents. The latter are calculated to order ε^3 of the ε expansion (three-loop approximation). Submitted to **Acta Physica Slovaca**.

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In recent years, considerable progress has been achieved in the understanding of intermittency and anomalous scaling of fluid turbulence. The crucial role in these studies was played by a simple model of a passive scalar quantity advected by a random Gaussian field, white in time and self-similar in space, the so-called Kraichnan's rapid-change model [1]. There, for the first time the existence of anomalous scaling was established on the basis of a microscopic model [2] and the corresponding anomalous exponents were calculated within controlled approximations [3–6] and a systematic perturbation expansion in a formal small parameter [7]. Detailed review of the recent theoretical research on the passive scalar problem and the bibliography can be found in [8].

The advection of a passive scalar field $\theta(x) \equiv \theta(t, \mathbf{x})$ is described by the stochastic equation

$$\partial_t \theta + (v_i \partial_i)\theta = \nu_0 \Delta\theta + f, \tag{1}$$

where $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, ν_0 is the molecular diffusivity coefficient, Δ is the Laplace operator, $\mathbf{v}(x) \equiv \{v_i(x)\}$ is the transverse (owing to the incompressibility) velocity field, and $f \equiv f(x)$ is an artificial Gaussian scalar noise with zero mean and correlator

$$\langle f(x) f(x') \rangle = \delta(t-t')\, C(r/L), \quad r = |\mathbf{x} - \mathbf{x}'|. \tag{2}$$

The parameter L is an integral scale related to the scalar noise and $C(r/L)$ is some function finite as $L \to \infty$. Without loss of generality, one can take $L = \infty$ and $C(0) = 1$.

In the real problem, the field $\mathbf{v}(x)$ satisfies the Navier–Stokes equation. In the rapid-change model it obeys a Gaussian distribution with zero mean and correlator

$$\langle v_i(x) v_j(x') \rangle = D_0 \frac{\delta(t-t')}{(2\pi)^d} \int d\mathbf{k}\, P_{ij}(\mathbf{k})\, k^{-d-\varepsilon} \exp[\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')], \tag{3}$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector, $k \equiv |\mathbf{k}|$, $D_0 > 0$ is an amplitude factor, d is the dimensionality of the $\mathbf{x}$ space and $0 < \varepsilon < 2$ is a parameter with the real ("Kolmogorov") value $\varepsilon = 4/3$. The infrared (IR) regularization is provided by the cut-off in the integral (3) from below at $k \simeq m$, where $m \equiv 1/\ell$ is the reciprocal of another integral scale ℓ; the precise form of the cut-off is unessential. The relation $D_0/\nu_0 = \Lambda^\varepsilon$ defines the characteristic ultraviolet (UV) momentum scale Λ.

The issue of interest is, in particular, the behaviour of the equal-time structure functions $S_n(r) = \big\langle [\theta(t,\mathbf{x}) - \theta(t,\mathbf{x}')]^n \big\rangle$, $r = |\mathbf{x} - \mathbf{x}'|$, in the inertial range $\Lambda \gg 1/r \gg m$. (With our assumptions, the odd structure functions vanish; they become nontrivial if the correlation function $\langle vf \rangle$ is nonzero or a constant gradient of the scalar field is imposed.) If we neglect the advection, that is, the nonlinearity in (1), these functions can be easily calculated: $S_n(r) = C_n \nu_0^{-n} r^{2n}$ with some constants C_n. In the full nonlinear problem, they become dependent on two additional variables Λr and mr:

$$S_n(r) = \nu_0^{-n} r^{2n} F_n(\Lambda r, mr). \tag{4}$$

The analysis by Refs. [3,4] shows that the behaviour of the functions F_n in (4) at $\Lambda r \to \infty$, $mr \to 0$ (inertial range) is nontrivial:

$$S_n(r) \simeq C_n \nu_0^{-n} r^{2n} (\Lambda r)^{-\alpha_n} (mr)^{-\beta_n}, \tag{5}$$

that is, the dependence on the both scales persists. Moreover, the exponent β_n is a nonlinear function of n, the phenomenon referred to as "anomalous scaling" in the theory of turbulence.

Relations (5) and explicit expressions for the exponents α_n and β_n up to the first order in $1/d$ and ε were derived in [3,4]. Within the "zero-mode approach" to Kraichnan's model, developed in those papers, anomalous exponents are related to the zero modes (unforced solutions) of the closed exact equations satisfied by the equal-time correlations.

In Ref. [7] and subsequent papers [9–17], the field theoretic renormalization group (RG) and operator product expansion (OPE) were applied to the rapid-change model and its descendants. In the RG approach, anomalous scaling emerges as a consequence of the existence in the model of composite operators with negative scaling dimensions, identified with the anomalous exponents. This allows one to construct a systematic perturbation expansion for the anomalous exponents, analogous to the famous ε expansion in the RG theory of critical behaviour, and to calculate the exponents to the second [7,9,10] and third [13,14] orders. For passively advected *vector* fields, any calculation of the exponents for higher-order correlations calls for the RG techniques already in the lowest-order approximation [15–17]. Besides the calculational efficiency, an important advantage of the RG approach is its relative universality: it is not related to the aforementioned "solvability" of the rapid-change model and can also be applied to the case of finite correlation time or non-Gaussian advecting field [12].

Below we briefly review the main ideas and results of the RG approach to the rapid-change model; detailed exposition and more references can be found in [7,12,14,17].

The solution proceeds in two stages. In the first stage, the RG equation is employed to find the asymptotic behaviour of the functions F_n in (4) with respect to their first argument ($\Lambda r \to \infty$) at fixed mr. This gives

$$S_n(r) \simeq \nu_0^{-n} r^{2n} (\Lambda r)^{-\alpha_n} F_n(mr). \tag{6}$$

The total power of r, namely $2n - \alpha_n$, is (up to the minus sign) the critical dimension of the quantity on the left-hand side. However, the form of the function $F_n(mr)$ cannot be determined by the RG equation.

Expression (4) is the special example of the general property established by the RG method for problem (1)–(3): the property of the IR scale invariance. Similar asymptotic expressions can be obtained for all correlation functions; in particular, for the equal-time pair correlation functions $S_{nk}(r) = \langle \Phi_n(x)\Phi_k(x')\rangle$ of the monomials $\Phi_n = [\partial_i\theta(x)\partial_i\theta(x)]^n$ one obtains:

$$S_{nk}(r) \simeq \nu_0^{-n-k} (\Lambda r)^{-\gamma_{nk}} F_{nk}(mr). \tag{7}$$

In the RG technique, the exponents α_n and γ_{nk} in (6), (7) are calculated in the form of series in the parameter ε from (2), which is therefore the analogue of the quantity $\varepsilon = 4 - d$ in the ϕ^4 model of critical behaviour.

From the general RG viewpoint, model (1)–(3) is simpler than the conventional ϕ^4 model in a few respects: the coordinate of the fixed point is exactly determined by the one-loop approximation, renormalization of the primary field $\theta(x)$ and its powers $\theta^n(x)$ ("composite operators" in the field theoretic language) is absent; the "mass" m is not renormalized. Therefore the exponents α_n in (6) are found exactly, $\alpha_n = n\varepsilon$, that is, they have no corrections of order ε^2, ε^3 and so on. However the exponents γ_{nk}, given by the sums $\gamma_{nk} = \Delta_n + \Delta_k$ of the critical dimensions Δ_n of the composite operators Φ_n, have nontrivial ε expansion. The knowledge of these dimensions allows one to find the asymptotic form of the mean values of the Φ_n at $m/\Lambda \to 0$:

$$\langle \Phi_n \rangle \simeq C_n \nu_0^{-n} (m/\Lambda)^{\Delta_n}. \tag{8}$$

Relation (8) gives a contribution in $\langle \Phi_n \rangle$ that is nonanalytic in m/Λ; this contribution prevails if $\Delta_n < 0$ (the operator is "dangerous"), which is indeed the case: in the first order in ε one obtains $\Delta_n = -2n(n-1)\varepsilon/(d+2)$, cf. [3]. Strictly speaking, because of the mixing in the renormalization the monomial Φ_n becomes a finite linear combination of the "scaling" operators with definite dimensions Δ_k with $k \le n$; in representations (7) and (8) we show only the leading term with the maximum $|\Delta_k|$.

The scaling functions $F_n(mr)$, $F_{nk}(mr)$ in (6), (7) can be calculated as series in ε, but such representations become useless if the behaviour of these functions at $mr \to 0$ is studied: the actual expansion parameter then becomes $\varepsilon \ln(mr)$ rather than ε. Summation of such "large IR logarithms," in contrast with the "large UV logarithms" with $\ln(\Lambda r)$, lies beyond the scope of the RG method in the narrow sense of the word, because the forms of the scaling functions are not determined by the plain RG equations. Like in the theory of critical behaviour, the behaviour at $mr \to 0$ can be extracted from the general solution of the RG equations by means of the operator product expansion (OPE). According to the OPE, the product of two (renormalized) composite operators $O_1(x_1)O_2(x_2)$ (for example, θ^n or renormalized analogues of the monomials Φ_n) at $t_1 = t_2 = t$, $\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2 \to 0$ and the fixed "centre of mass" $\mathbf{x} \equiv (\mathbf{x}_1 + \mathbf{x}_2)/2 = \text{const}$ is represented in the form

$$O_1(x_1)O_2(x_2) = \sum_\alpha C_\alpha(\mathbf{r}) O_\alpha(\mathbf{x}, t), \tag{9}$$

where C_α are coefficients analytic in $(mr)^2$ and the summation, in general case, runs over all local composite operators $O_\alpha(x)$ with definite critical dimensions Δ_α. Correlation functions like (6), (7) are obtained by averaging relations (9); the mean values $\langle O_\alpha(x)\rangle \propto m^{\Delta_\alpha}$ appear on the right-hand sides, they give rise to the contributions of the form $(mr)^{\Delta_\alpha}$ in the scaling functions $F_n(mr)$, $F_{nk}(mr)$. The leading terms at $mr \to 0$ are determined by the operator with the minimum possible value of Δ_α. In conventional models like ϕ^4 the leading term is related to the simplest operator $O_\alpha(x) = 1$ with $\Delta_\alpha = 0$. However, in model (1)–(3) the critical dimensions of all operators Φ_n are negative (see above), and they determine the leading terms for $mr \to 0$. Although the operators containing fields θ without the derivative ∂ also have negative dimensions, they do not contribute to the OPE for the quantities like (6), (7) due to the invariance of the latter with respect to the shift $\theta \to \theta + \mathrm{const}$ of the field $\theta(x)$.

If the operator product expansion for any given function (6) or (7) included contributions from all the operators Φ_n, we would have to sum all of them in order to obtain the behaviour for $mr \to 0$, because the spectrum of their dimensions is not bounded from below (the most dangerous operator does not exist). This is the case in more realistic models of fully developed turbulence; see [18–20]. Fortunately, this problem does not exist in model (1)–(3) due to the linearity of equation (1) in $\theta(x)$: one can show that the number of fields θ in any composite operator $O_\alpha(x)$ that enters the right-hand side of representation (9) cannot exceed the total number of fields on the left-hand side. We thus arrive at the following expressions for quantities (6), (7):

$$S_n(r) \simeq \nu_0^{-n} r^{2n} (\Lambda r)^{-n\varepsilon} \sum_{s\le n} C_{n,s} (mr)^{\Delta_s}, \tag{10}$$

$$S_{nk}(r) \simeq \nu_0^{-n-k} (\Lambda r)^{-\Delta_n-\Delta_k} \sum_{s\le n+k} C_{nk,s} (mr)^{\Delta_s} \tag{11}$$

with coefficients $C_{n,s}$, $C_{nk,s}$ dependent on ε and d and corrections of the form $(mr)^{2+O(\varepsilon)}$. Since the dimensions Δ_s are negative and decrease as s increases, the leading asymptotic term in (10), (11) is given by the contribution with the maximum s, that is, with $s = n$ in (10) and $s = n + k$ in (11):

$$S_n(r) \simeq C_n \nu_0^{-n} r^{2n} (\Lambda r)^{-n\varepsilon} (mr)^{\Delta_n}, \tag{12}$$

$$S_{nk}(r) \simeq C_{nk} \nu_0^{-n-k} (\Lambda r)^{-\Delta_n-\Delta_k} (mr)^{\Delta_{n+k}}. \tag{13}$$

The critical dimensions are calculated as series of the form $\Delta_n = \sum_{k=1}^{\infty} \varepsilon^k \Delta_n^{(k)}$; the first-order term was already given above. We have calculated the dimensions Δ_n in the second [7] and third [13,14] orders of the ε expansion. The second-order result has the form:

$$\Delta_n^{(2)} = \frac{n(n-1)}{(d-1)(d+2)^3(d+4)^2} \Big[-4(d+1)(d+4)^2 + $$
$$+3(d-1)(d+2)(d+4)(d+2n)h(d) - 4(d+1)(d+2)(d+3n-2)h(d+2) \Big], \tag{14}$$

where $h(d) \equiv F(1,1;d/2+2;1/4)$ and $F(\cdots)$ is the hypergeometric series. Simpler expressions are obtained for integer d, in particular, $h(2) = 8[1 - 3\ln(4/3)]$, $h(3) = 10(\pi\sqrt{3} - 16/3)$, while for the other integer d analogous expressions can be obtained from the recurrent relation $3h(d) + (d+2)h(d+2)/(d+4) = 4$. The third-order coefficient is presented in [13,14].

From the above formulas one obtains $\Delta_1^{(2)} = 0$ in agreement with the exact result $\Delta_1 = 0$. The formal proof of the latter is based on certain Schwinger equation, which has the meaning of the conservation law for the "energy" $\theta^2(x)$; see [7]. This means that the second-order structure function S_2 is not anomalous in agreement with the well-known exact solution obtained in [1].

We note that the family of operators Φ_n is "closed with respect to the fusion" in the sense that the leading term of the correlation function $\langle \Phi_n \Phi_m \rangle$ is given by the operator Φ_{n+m} from the same family with the summed index $n + m$. This fact along with the inequality $\Delta_n + \Delta_m > \Delta_{n+m}$, which is obvious from the explicit expressions for Δ_n, can be interpreted as the statement that the correlations of the local dissipation rate in the model (1)–(3) exhibit multifractal behaviour, cf. [21,22].

An important issue that can be discussed on the example of the rapid-change model is that of the nature and convergence of ε series in models of turbulence and the possibility of their extrapolation to finite values of $\varepsilon \sim 1$. The knowledge of the three terms of the ε expansion in model (1)–(3) allows one to discuss its convergence properties

and to obtain improved predictions for finite ε in reasonable agreement with the existing nonperturbative results: analytical and numerical solutions of the zero-mode equations [4,5] and numerical experiments [23,24].

The RG and OPE approach presented above can be generalized to the cases where compressibility, anisotropy or finite correlation time are present [10–14] and the passive advection of vector (e.g., magnetic) fields [15–17].

Let us conclude with a brief discussion of the RG and OPE approach to a more realistic model of fluid turbulence: the stirred Navier–Stokes equation; see [18–20]. Dangerous operators in that model are absent in the ε expansions and can appear only at finite values of ε. This means that they can be reliably identified only if their dimensions are derived *exactly* with the aid of Schwinger equations or Galilean symmetry. Due to the nonlinear nature of the problem, they enter the corresponding OPE's as infinite families whose spectra of dimensions are not bounded from below, and in order to find the small-mr behavior one has to sum up all their contributions in the representations like (12), (13). The needed summation of the most singular contributions, related to the operators of the form v^n with known dimensions, was performed in [18] using the so-called infrared perturbation theory for the case of the different-time pair correlation functions. It has revealed their strong dependence on m, which physically can be explained by the well-known "sweeping effects." This demonstrates that, contrary to the existing opinion [25,26], the sweeping effects can be properly described within the RG approach, but one should combine the RG and OPE techniques and go beyond the plain ε expansions. Analysis of the m dependence of the Galilean invaiant objects like the structure functions requires the explicit construction of all dangerous invariant scalar operators, exact calculation of their critical dimensions, and summation of their contributions in the corresponding OPE. This is clearly not a simple problem and it requires considerable improvement of the present techniques.

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[1] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[2] R. H. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
[3] D. Bernard, K. Gawędzki, and A. Kupiainen, Phys. Rev. E **54**, 2564 (1996).
[4] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **52**, 4924 (1995).
[5] A. Pumir, Phys. Rev. E **57**, 2914 (1998).
[6] B. I. Shraiman and E. D. Siggia, Phys. Rev. Lett. **77**, 2463 (1996).
[7] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, Phys. Rev. E **58**, 1823 (1998).
[8] G. Falkovich, K. Gawędzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).
[9] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, Theor. Math. Phys. **120**, 1074 (1999).
[10] L. Ts. Adzhemyan and N. V. Antonov, Phys. Rev. E **58**, 7381 (1998).
[11] N. V. Antonov and J. Honkonen, Phys. Rev. E **63**, 036302 (2001).
[12] N. V. Antonov, Phys. Rev. E **60**, 6691 (1999).
[13] L. Ts. Adzhemyan, N. V. Antonov, V. A. Barinov, Yu. S. Kabrits, and A. N. Vasil'ev, Phys. Rev. E **63**, 025303(R) (2001);
[14] L. Ts. Adzhemyan, N. V. Antonov, V. A. Barinov, Yu. S. Kabrits, and A. N. Vasil'ev, Phys. Rev. E **64**, 019901(E) (2001); E **64**: 056306 (2001).
[15] N. V. Antonov, A. Lanotte, and A. Mazzino, Phys. Rev. E **61**, 6586 (2000).
[16] L. Ts. Adzhemyan, N. V. Antonov, A. Mazzino, P. Muratore Ginanneschi, and A.V.Runov, *Europhys. Lett.* **55**: 801 (2001).
[17] L. Ts. Adzhemyan, N. V. Antonov, and A. V. Runov, Phys. Rev. E **64**: 046310 (2001)
[18] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, Zh. Éksp. Teor. Fiz. **95**, 1272 (1989) [Sov. Phys. JETP **68**, 733 (1989)].
[19] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasil'ev, Usp. Fiz. Nauk, **166**, 1257 (1996) [Phys. Usp. **39**, 1193 (1996)].
[20] L. Ts. Adzhemyan, N. V. Antonov, and A. N. Vasiliev, *The Field Theoretic Renormalization Group in Fully Developed Turbulence* (Gordon & Breach, London, 1999).
[21] B. Duplantier and A. Ludwig, Phys. Rev. Lett. **66**, 247 (1991).
[22] G. L. Eyink, Phys. Lett. A **172**, 355 (1993).
[23] U. Frisch, A. Mazzino, A. Noullez, and M. Vergassola, Phys. Fluids **11**, 2178 (1999).
[24] A. Mazzino and P. Muratore-Ginanneschi, Phys. Rev. E **63**, 015302(R) (2001).
[25] S. Chen and R. H. Kraichnan, Phys. Fluids A **1**, 2019 (1989).
[26] S. L. Woodruff, Phys. Fluids A **6**(9), 3051 (1994).